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Determining whether a finite group of outer automorphisms of a free group is geometric

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Abstract

In this article we give necessary and sufficient conditions for a given finite group of outer automorphisms to be induced by the action of a group of orientation-preserving homeomorphisms on the fundamental group of a punctured surface. When the group is abelian, necessary and sufficient conditions can also be given in the absence of orientability assumptions. These properties are formulated in terms of the finite automorphism groups which project into the given outer automorphism group: each non-trivial automorphism in any such group can fix at most a cyclic subgroup of the fundamental group.

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1. Introduction

A group of homeomorphisms of a surface induces a group of outer automorphisms of the fundamental group which is called *geometric*. In this paper, we concern ourselves with deciding which groups of outer automorphisms are geometric.

The fundamental group of a compact surface S can be presented as

$$\pi_1(S) = \left\langle s_1, \ldots, s_k, u_1, t_1, \ldots, u_g, t_g \,|\, s_1 \cdots s_k \prod_{i=1}^{g} [u, t_i] = 1 \right\rangle$$

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or

$$\pi_1(S) = \left\langle s_1, \ldots, s_k, v_1, \ldots, v_g \mid s_1 \cdots s_k \prod_{i=1}^{g} v_i^2 = 1 \right\rangle$$

for S respectively orientable and non-orientable, where the s_i correspond to the boundary components of S, if any. Between 1927 and 1939, Dehn, Nielsen [10], and Mangler [11] established the following result:

Theorem. *An automorphism of $\pi_1(S)$ is induced by a homeomorphism of S if and only if there is a permutation σ of $\{1, \ldots, k\}$, exponents $\varepsilon_i \in \{-1, 1\}$, and words $w_i \in \pi_1(S)$ such that s_i maps to $w_i s_{\sigma(i)}^{\varepsilon_i} w_i^{-1}$. Furthermore, with $\mathrm{o}(w_i)$ defined to be -1 if there are an odd number of the generators v_j in word w_i and 1 otherwise, it is necessary that $\mathrm{o}(w_i)\varepsilon_i$ be the same for all i.*

It follows that every group of automorphisms of a closed surface group is geometric. It is not guaranteed, though, that the realization is faithful in the sense that the only homeomorphism homotopic to the identity is trivial. For the free fundamental groups of punctured surfaces, these deciding conditions are difficult to check when the group is presented as $\pi_1(S) = \langle x_1 \ldots x_n \rangle$. These conditions does not always hold, as Stallings [13] showed in 1982 with a collection of non-geometric automorphisms.

In 1992, Bestvina and Handel [1] proved that an outer automorphism which preserves a non-trivial word up to cyclic permutation and has every non-trivial power irreducible is realizable by a surface homeomorphism (necessarily non-periodic). In 1993, Dicks and Ventura [3] proved that every periodic irreducible outer automorphism is geometric. In this paper, only periodic outer automorphisms are examined, though the irreducibility requirement is dropped. The result of Dicks and Ventura follows as a corollary to Theorem 4.8.

Though it is conceivable that a geometric group might not have a faithful realization, Zieschang ([14, p. 6]) extended a result of Nielsen to show that this can never be the case for a finite group of outer automorphisms of a finitely generated free group. Kerckhoff [6] extended this result to the more difficult case of closed-surface-group automorphisms in 1983. Thus, in this paper we are justified in working exclusively with periodic maps of graphs and surfaces.

By the Realization Theorem [15,2], every finite group $\mathbf{G}$ of outer automorphisms of a finitely generated free group can be realized by a group of homeomorphisms of a graph. Given such a graph and homeomorphisms, Krstić [8] in 1989 showed how to generate all the finitely many reduced graphs (no invariant forests) that realize $\mathbf{G}$. From here, one can check certain properties at each of the vertices and edges of each graph, using the work of Los and Nitecki [9], to determine if $\mathbf{G}$ is geometric. In this paper, we follow these steps to obtain necessary and sufficient conditions for $\mathbf{G}$ to be geometric, based on the finite groups of automorphisms which project into $\mathbf{G}$ and the fixed subgroups of those automorphisms.

In 1995, Khramtsov [7] obtained algorithms to find the finite subgroups of an arbitrary almost free group and to find the fixed subgroup of an arbitrary finite group of free-group automorphisms. Since the collection of all automorphisms that project into **G** is almost free, it follows that the conditions given in this paper can be checked by an algorithm.

2. Foundations

We begin by establishing some notation and by detailing the relationship between groups of automorphisms of a graph and the (outer) automorphisms they realize.

Definition 2.1. By $\mathrm{Aut}(F)$ we denote the group of automorphisms of the free group F on n generators. For $f \in \mathrm{Aut}(F)$, the fixed subgroup of f is $\mathrm{Fix}(f) = \{x \in F \mid f(x) = x\}$. For $x \in F$, $\mathrm{Inn}_x \in \mathrm{Aut}(F)$ denotes the inner automorphism which takes each $y \in F$ to xyx^{-1}. The group of inner automorphisms $\mathrm{Inn}(F) = \{\mathrm{Inn}_x \mid x \in F\}$ is a normal subgroup of $\mathrm{Aut}(F)$, so we can define $\mathrm{Out}(F)/\mathrm{Aut}(F)/\mathrm{Inn}(F)$, the group of outer automorphisms with quotient map:

$$\Phi : \mathrm{Aut}(F) \to \mathrm{Out}(F).$$

Notice that if $H \subset \mathrm{Aut}(F)$ is a finite subgroup, then $\Phi|_H : H \to \mathrm{Out}(F)$ is injective because non-trivial inner automorphisms are of infinite order.

Definition 2.2. A graph Γ has vertices $V(\Gamma)$ and (directed) edges $E(\Gamma)$. Edge e begins at $i(e) \in V(\Gamma)$, ends at $t(e) \in V(\Gamma)$, and spans a subgraph $|e| \subset \Gamma$ called a *geometric edge*. The inverse $\bar{e} \in E(\Gamma)$ spans the same subgraph but has the opposite orientation. The star of $v \in V(\Gamma)$ is star $(v) = \{e \in E(\Gamma) \mid i(e) = v\}$ and the valence of v is the number of edges in $\mathrm{star}(v)$.

Definition 2.3. A path $\gamma = e_1 \ldots e_k$ is a concatenated sequence of edges with $t(e_i) = i(e_i + 1)$ which is reduced if $e_i \neq \bar{e}_{i+1}$ for $i = 1, \ldots, k$. The trivial path at $v \in V(\Gamma)$ is denoted 1_v.

Definition 2.4. A graph automorphism $\hat{g} : \Gamma \to \Gamma$ is a homeomorphism which takes vertices to vertices and is affine on edges. If $\gamma = e_1 \ldots e_k$ is a path then the image $\hat{g}(\gamma) = \hat{g}(e_1) \cdots \hat{g}(e_k)$ is reduced if and only if γ is reduced. If Γ is connected than each $\hat{g} \in \mathrm{Aut}(\Gamma)$ (the group of all graph automorphisms of Γ) can be uniquely described by the bijection of $E(\Gamma)$ it induces. The order of $\hat{g}$ is the least integer $k \geqslant 1$ such that $\hat{g}^k = \hat{id}$ (the identity graph automorphism). Note that every subgroup $\hat{G} \subseteq \mathrm{Aut}(\Gamma)$ is finite. The stabilizer under $\hat{G}$ of an edge or vertex x is $\mathrm{stab}(x) = \{\hat{g} \in \hat{G} \mid \hat{g}(x) = x\}$.The orbit of $e \in E(\Gamma)$ under $\hat{G}$ is $\hat{G}e = \{\hat{g}(e) \mid \hat{g} \in \hat{G}\}$. Group $\hat{G}$ is reduced if the subgraph spanned by $\hat{G}e$ (denoted $|\hat{G}e|$) contains a simple circuit for each $e \in E(\Gamma)$. Equivalently, $\hat{G}$ is reduced if every edge $e \in E(\Gamma)$ belongs to a simple circuit in $|\hat{G}e|$. We can ensure that $\hat{G}$ is non-edge-reversing by subdividing any edge in Γ which maps to its inverse.

Definition 2.5. For a path γ in Γ, $[\gamma]$ denotes the equivalence class of all paths in Γ homotopic to γ rel endpoints. By simple homotopy theory it can be seen that in every $[\gamma]$ there is a unique reduced representative.

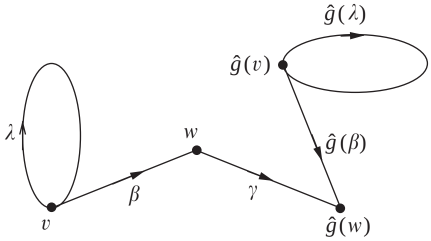

Fig. 1. A subgraph of graph Γ.

Definition 2.6. A graph automorphism $\hat{g} \in \mathrm{Aut}(\Gamma)$ realizes a free-group automorphism in the following manner:

- $M : F \to \pi_1(\Gamma, v)$ is a *marking* at base vertex v. We fix a marking for every graph Γ. To get a new vertex w to act as the base, we use a path β from v to w:
- $\beta^* : \pi_1(\Gamma, v) \to \pi_1(\Gamma, w)$ takes each $[\lambda] \in \pi_1(\Gamma, v)$ to $[\bar{\beta}\lambda\beta]$.
- $\hat{g}_* : \pi_1(\Gamma, w) \to \pi_1(\Gamma, \hat{g}(w))$ takes that $[\bar{\beta}\lambda\beta]$ to $[\hat{g}(\bar{\beta})\hat{g}(\lambda)\hat{g}(\beta)]$.
- $M^{-1}\bar{\beta}^*\bar{\gamma}^* : \pi_1(\Gamma, \hat{g}(w)) \to F$ brings everything back to F with a path γ from w to $\hat{g}(w)$ (Fig. 1).

The resulting automorphism will be written as $A(\hat{g}, \beta, \gamma) = (M_{\beta\gamma})^{-1}\hat{g}_* M_\beta \in \mathrm{Aut}(F)$ where M_β denotes $\beta^* M$. Any mention of $A(\hat{g}, \beta, \gamma)$ will imply that γ is a path from $t(\beta)$ to $\hat{g}(t(\beta))$. However, after we establish the next few results, we will only concern ourselves with automorphisms realized at fixed vertices w. We will write $A(\hat{g})$ for $A(\hat{g}, \beta, 1_w)$ when $\hat{g} \in \mathrm{stab}(w)$ and the path β from v to w is clear. Similarly, $A(\hat{H})$ will denote $\{A(\hat{h}) \mid \hat{h} \in \hat{H}\}$ for $\hat{H} \subseteq \mathrm{stab}(w)$.

Lemma 2.7.

(1) $A(\hat{g}, \beta, \gamma) = A(\hat{g}, \alpha, \bar{\alpha}\beta\gamma\hat{g}(\bar{\beta}\alpha))$ *for any path* α *at* v.
(2) $A(\hat{f}, \beta, \delta)A(\hat{g}, \beta, \gamma) = A(\hat{f}\hat{g}, \beta, \delta\hat{f}(\gamma))$
(3) $A(\hat{id}, \beta, \lambda) = \mathrm{Inn}_x$ *for which* $M_\beta(x) = [\lambda]$.

Proof. By straight calculation we can establish:
(i) $\hat{g}_*(\beta^*)^{-1} = (\hat{g}(\beta)^*)^{-1}\hat{g}_*$,
(ii) $M_{\alpha\beta} = \beta^* M_\alpha$, and
(iii) $(M_{\alpha\beta})^{-1} = (M_\alpha)^{-1}(\beta^*)^{-1}$.

(1) $A(\hat{g}, \beta, \gamma) = (M_{\beta\gamma})^{-1}\hat{g}_* M_\beta = (M_{\alpha\bar{\alpha}\beta\gamma})^{-1}\hat{g}_*((\bar{\beta}\alpha)^*)^{-1}(\bar{\beta}\alpha)^* M_\beta \overset{\text{(i),(ii)}}{=} (M_{\alpha\bar{\alpha}\beta\gamma})^{-1} (\hat{g}(\bar{\beta}\alpha)^*)^{-1}\hat{g}_* M_{\beta\bar{\beta}\alpha} \overset{\text{(iii)}}{=} (M_{\alpha\bar{\alpha}\beta\gamma\hat{g}(\bar{\beta}\alpha)})^{-1}\hat{g}_* M_\alpha = A(\hat{g}, \alpha, \bar{\alpha}\beta\gamma\hat{g}(\bar{\beta}\alpha))$.

(2) $A(\hat{f}, \beta, \delta)A(\hat{g}, \beta, \gamma) = (M_{\beta\delta})^{-1}\hat{f}_* M_\beta (M_{\beta\gamma})^{-1}\hat{g}_* M_\beta \overset{\text{(iii)}}{=} (M_{\beta\delta})^{-1}\hat{f}_* M_\beta (M_\beta)^{-1}(\gamma^*)^{-1} \hat{g}_* M_\beta \overset{\text{(i)}}{=} (M_{\beta\delta})^{-1}(\hat{f}(\gamma)^*)^{-1}\hat{f}_*\hat{g}_* M_\beta \overset{\text{(iii)}}{=} (M_{\beta\delta\hat{f}(\gamma)})^{-1}(\hat{f}\hat{g})_* M_\beta = A(\hat{f}\hat{g}, \beta, \delta\hat{f}(\gamma))$.

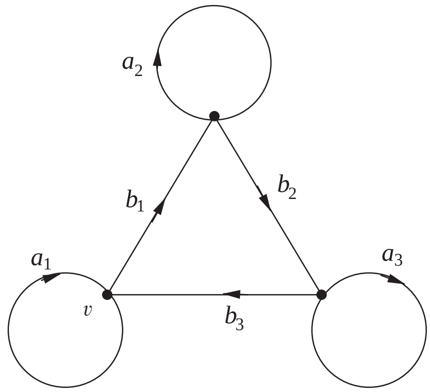

Fig. 2. Graph Γ.

(3) If $y \in F$ and $M_\beta(y) = [\alpha] \in \pi_1(\Gamma, t(\beta))$ then $A(\hat{id}, \beta, \lambda)(y) = (M_{\beta\lambda})^{-1}\hat{id}_* M_\beta(y) \overset{\text{(iii)}}{=} (M_\beta)^{-1}(\lambda^*)^{-1}[\alpha] = (M_\beta)^{-1}[\lambda\alpha\bar{\lambda}] = xyx^{-1} = \mathrm{Inn}_x(y)$. $\square$

Corollary 2.8. *The following two subgroups of* $\mathrm{Aut}(F)$ *are equal*: $\{\mathrm{Inn}_x\, A(\hat{g}, \beta, \gamma) \mid x \in F\} = \{A(\hat{g}, 1_v, \delta) \mid \delta$ *is a path from* v *to* $\hat{g}(v)\}$.

Proof. If $x \in F$ and $M_\beta(x) = [\lambda]$ then $\mathrm{Inn}_x\, A(\hat{g}, \beta, \gamma) = A(\hat{id}, \beta, \lambda)A(\hat{g}, \beta, \gamma) = A(\hat{g}, \beta, \lambda\gamma) = A(\hat{g}, 1_v, \beta\lambda\gamma\hat{g}(\bar{\beta}))$ by the above lemma. Any path δ from v to $\hat{g}(v)$ can be written as $\beta\lambda\gamma\hat{g}(\bar{\beta})$ with $\lambda = \bar{\beta}\delta\hat{g}(\beta)\bar{\gamma}$. $\square$

This proves that $\Phi(A(\hat{g}, \beta, \gamma)) \in \mathrm{Out}(F)$ is independent of both β and γ, so the following is well-defined.

Definition 2.9. The outer automorphism realized by $\hat{g} \in \mathrm{Aut}(\Gamma)$ is denoted $\mathbf{O}(\hat{g}) = \Phi(A(\hat{g}, \beta, \gamma)) \in \mathrm{Out}(F)$ for any path β from v to w and path γ from w to $\hat{g}(w)$. For $\hat{G} \subseteq \mathrm{Aut}(\Gamma)$, $\mathbf{O}(\hat{G}) = \{\mathbf{O}(\hat{g}) \mid \hat{g} \in \hat{G}\} \subset \mathrm{Out}(F)$.

Theorem 2.10 (*The Realization Theorem*). *For each finite group* $\mathbf{G} \subset \mathrm{Out}(F)$ *there is a compact graph* Γ *and* $\hat{G} \subseteq \mathrm{Aut}(\Gamma)$ *such that* $\mathbf{O}(\hat{G}) = \mathbf{G}$.

The proofs in both [2,18] are based on applying a deep result of Karass et al. [5] to the almost free group $\Phi^{-1}(\mathbf{G})$.

Example 2.11. Let f be the automorphism of $F = \langle x_1, x_2, x_3, x_4\rangle$ which takes x_1 to x_2, x_2 to x_3, x_3 to $x_4x_1x_4^{-1}$, and x_4 to x_4. Since f^3 is an inner automorphism, $\Phi(f)$ generates a finite group of outer automorphisms.

Now consider the following example (Fig. 2).

$\hat{f} \in \mathrm{Aut}(\Gamma)$ takes $a_i \to a_{i+1} \pmod 3$ and $b_i \to b_{i+1} \pmod 3$.

With the marking $M : F \to \pi_1(\Gamma, v)$ which takes x_1 to $[a_1]$, x_2 to $[b_1 a_2 \bar{b}_2]$, x_3 to $[b_1 b_2 a_3 \bar{b}_2 \bar{b}_1]$, and x_4 to $[b_1 b_2 b_3]$, our original automorphism f is realized $f = A(\hat{f}, 1_v, b_1)$.

The following proposition shows how finite subgroups of the automorphisms correspond with fixed vertices in the graph.

Proposition 2.12. *Let $\hat{G} \subseteq \mathrm{Aut}(\Gamma)$ be a non-edge-reversing group of graph automorphisms and $G = \Phi^{-1}(\mathbf{O}(\hat{G})) \subseteq \mathrm{Aut}(F)$ be the induced group. A subgroup $H \subset G$ is finite if and only if there exists a subgroup of a vertex stabilizer $\hat{H} \subseteq \mathrm{stab}(w) \subseteq \hat{G}$ and a path β from v to w such that $H = A(\hat{H})$.*

Proof. Let $H = \{h_i \mid i = 1, \ldots, k\} \subset \Phi^{-1}(\mathbf{G})$. By Corollary 2.8 there exists $\{\hat{h}_i \mid i = 1, \ldots, k\} \subseteq \hat{G}$ and $\{\gamma_i$ a path from v to $\hat{h}_i(v) \mid i = 1, \ldots, k\}$ such that $h_i = A(\hat{h}_i, 1_v, \gamma_i)$ for $i = 1, \ldots, k$. Let T be the tree which universally covers Γ, and $v' \in V(T)$ be a lift of the basepoint v. If we lift γ_i to γ_i' in T starting at v', then the endpoint $t(\gamma_i')$ uniquely defines a lift $(\hat{h}_i(v))'$ of $\hat{h}_i(v)$. We can define automorphisms $\hat{h}_i' : T \to T$ which map v' to $\hat{h}_i(v)'$ and cover $\hat{h}_i$. By Lemma 2.7 we can see that the map $\hat{h}_i \to \hat{h}_i'$ is a homomorphism. It follows that the image of $\hat{H}$ is a finite group of non-edge-reversing maps of the tree T. It is well known (for instance [12, pp. 61–65]) that in this situation there must be a vertex $w' \in V(T)$ which is fixed by all the $\hat{h}_i'$.

Let β' be the path in T from v' to w'. By the uniqueness of reduced paths in a tree, $\beta' \hat{h}_i(\bar{\beta}')$, which goes from v' to $\hat{h}_i(v') = \hat{h}_i(v)'$, must reduce to γ_i'. Thus when we project β' down to a path β from v to some w in Γ, we see that $h_i = A(\hat{h}_i, 1_v, \gamma_i) = A(\hat{h}_i, 1_v, \beta \hat{h}_i(\bar{\beta})) = A(\hat{h}_i, \beta, \bar{\beta}\beta\hat{h}_i(\bar{\beta})\hat{h}_i(\beta)) = A(\hat{h}_i, \beta, 1_{t(\beta)})$ for $i = 1, \ldots, k$ by Lemma 2.7.

The converse is immediate by Lemma 2.7. $\square$

3. Geometric graph and outer automorphisms

In this section, we highlight the work of Los and Nitecki [9] in establishing necessary and sufficient conditions for a group of graph automorphisms to be realizable by homeomorphisms of a compact surface. With these, we establish necessary conditions for a finite group of outer automorphisms to have the same property.

Definition 3.1. If the graph Γ can be embedded as a strong deformation retract of a punctured surface S, we call it a *spine* for S (we will assume $\Gamma \subset S$). The group of homeomorphisms from S to itself is denoted $\mathrm{Homeo}(S)$, and if S is orientable the subgroup of orientation-preserving homeomorphisms is $\mathrm{Homeo}^+(S)$. If Γ is a spine of S and $\tilde{G} \subset \mathrm{Homeo}(S)$, then Γ is $\tilde{G}$-invariant if, for each $\tilde{g} \in \tilde{G}$, $\tilde{g}(\Gamma) = \Gamma$ and $\tilde{g}(V(\Gamma)) = V(\Gamma)$.

Theorem 3.2 (*Los and Nitecki [9]*). *For every compact connected punctured surface S and finite subgroup $\tilde{G} \subset \mathrm{Homeo}(S)$, there is a $\tilde{G}$-invariant spine.*

Definition 3.3. A finite group $\hat{G} \subseteq \mathrm{Aut}(\Gamma)$ is called *geometric* (resp. *orientably-geometric*) if there is a compact connected punctured surface S and group $\tilde{G} \subset \mathrm{Homeo}(S)$

(resp. $\tilde{G} \subset \mathrm{Homeo}^+(S)$) for which Γ is a $\tilde{G}$-invariant spine and $\tilde{G}|_\Gamma = \{\tilde{g}|_\Gamma$ for each $\tilde{g} \in \bar{G}\} = \hat{G} \subseteq \mathrm{Aut}(\Gamma)$.

If $\hat{G} \subseteq \mathrm{Aut}(\Gamma)$ is geometric and $w \in V(\Gamma)$, we shall see that any $\hat{g} \in \mathrm{stab}(w)$ must have certain properties. This motivates the following.

Definition 3.4. Given $\hat{G} \subseteq \mathrm{Aut}(\Gamma)$, $w \in V(\Gamma)$, and $\hat{g} \in \mathrm{stab}(w) \subseteq \hat{G}$, we say

(1) the map $\hat{g}$ is *equicyclic at w* if for any $e \in \mathrm{star}(w)$ and $k \in \mathbb{Z}$, $\hat{g}^k(e) = e$ implies $\hat{g}^k = \hat{id}$, that is, $\hat{g}$ acts freely on $\mathrm{star}(w)$.
(2) the map $\hat{g}$ is *bicyclic at w* if $\hat{g}^2 = \hat{id}$ and $\hat{g}$ fixes no more than two edges of $\mathrm{star}(w)$.

Observe that if $\hat{g}$ is equicyclic at *w*, then every power of $\hat{g}$ is also equicyclic at *w*. If $\hat{g}^2 = \hat{id}$ and $\hat{g} \in \mathrm{stab}(w)$ fixes no edges of $\mathrm{star}(w)$, then $\hat{g}$ is both bicyclic at *w* and equicyclic at *w*. We will say that $\hat{g}$ is *strictly bicyclic at w* if $\hat{g}$ is bicyclic at w but not equicyclic at *w*.

We will use the following notation for cyclic and dihedral groups:

$$C_m = C(r, m) = \langle r \mid r^m = 1 \rangle,$$

$$D_m = D(f, r, m) = \langle f, r \mid f^2 = r^m = 1,\ frf = r^{-1} \rangle,$$

where $m \geqslant 1$. The subgroup $C_m \subset D_m$ will always refer to that generated by *r*. Though C_2 and D_1 represent the same group, use of the latter will imply that an edge is fixed by the group. Often the trivial group will be written as C_1.

A group of graph automorphisms $\hat{G} \subseteq \mathrm{Aut}(\Gamma)$ is *geometrizable at* $w \in V(\Gamma)$ if either $\mathrm{stab}(w) = C(\hat{r}, m)$ where $\hat{r}$ is equicyclic at *w* or $\mathrm{stab}(w) = D(\hat{f}, \hat{r}, m)$ where every element of $C_m \subset D_m$ is equicyclic at *w* and all other elements are bicyclic at *w*.

Remark 3.5. If $\hat{G} \subseteq \mathrm{Aut}(\Gamma)$ is geometrizable at $w \in V(\Gamma)$ and $\mathrm{stab}(w) = \hat{D}(\hat{f}, \hat{r}, m)$, then every element $\hat{g} \in \mathrm{stab}(w)$ has the form $\hat{g} = \hat{f}^p \hat{r}^q$, $p \in \{0, 1\}$ and $q \in \{0, \ldots, m-1\}$. If $\hat{g}$ is strictly bicyclic at *w*, then $p = 1$. The product of two such elements $\hat{f}\hat{r}^q \hat{f}\hat{r}^k = \hat{r}^{k-q}$ is equicyclic at *w*.

Theorem 3.6 (*Los and Nitecki [9]*). *Suppose Γ has no vertices of valence* 2 *and $\hat{G} \subseteq$ $\mathrm{Aut}(\Gamma)$ is a subgroup. Then*

(1) *$\hat{G}$ is geometric if and only if*
 (a) *the group $\hat{G}$ is geometrizable at every $w \in V(\Gamma)$,*
 (b) *if $\hat{g}(e) = \bar{e}$ for $\hat{g} \in \hat{G}$ and $e \in E(\Gamma)$, then $\hat{g}^2 = \hat{id}$.*
(2) *$\hat{G}$ is orientably-geometric if and only if for each $w \in V(\Gamma)$, $\mathrm{stab}(w) = C(\hat{r}, m)$ for $\hat{r}$ equicyclic at w.*

When we insist that a group $\hat{G} \subseteq \mathrm{Aut}(\Gamma)$ be non-edge-reversing, it is often necessary to accept vertices of valence 2. Thus, in order to apply the previous result for our purposes, we shall use the following reformulation:

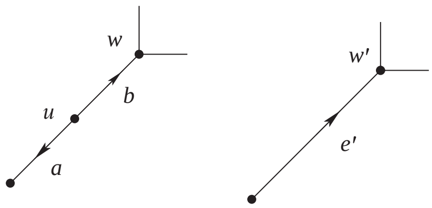

Fig. 3. The star of vertex u and the corresponding edge e'.

Theorem 3.7. *If $\hat{G} \subseteq \mathrm{Aut}(\Gamma)$ is reduced and non-edge-reversing then*

(1) *$\hat{G}$ is geometric if and only if $\hat{G}$ is geometrizable at every $w \in V(\Gamma)$.*
(2) *$\hat{G}$ is orientably-geometric if and only if for each $w \in V(\Gamma)$, $\mathrm{stab}(w) = C(\hat{r}, m)$ for $\hat{r}$ equicyclic at w.*

Proof. Let graph Γ' be Γ with all vertices of valence 2 removed, and $\hat{G}' \subseteq \mathrm{Aut}(\Gamma')$ be the group induced by $\hat{G} \subseteq \mathrm{Aut}(\Gamma)$. Since valence-2 vertex addition and removal can be done to the spine of any surface, $\hat{G}$ is (orientably) geometric if and only if $\hat{G}'$ is (orientably) geometric. Since conditions at each vertex $w \in V(\Gamma)$ of valence 3 or more are identical to those at the corresponding $w' \in V(\Gamma')$, $\hat{G}$ is geometrizable at w (resp. $\mathrm{stab}(w) = C(\hat{r}, m)$ for $\hat{r}$ equicyclic at w) if and only if $\hat{G}'$ is geometrizable at w' (resp. $stab(w') = C(\hat{r}, m)$ for $\hat{r}$ equicyclic at w').

Since $\hat{G}$ is reduced, there is a one-to-one correspondence between valence-2 vertices $u \in V(\Gamma)$ and reversed edges $e' \in E(\Gamma')$. More specifically, $\mathrm{star}(u) = \{a, b\}$ for which e' corresponds to $\bar{a}b$, and the corresponding endpoints $t(e') = w'$ and $t(b) = w$ are both of the same valence of three or more. Furthermore, there is an $\hat{r} \in \mathrm{stab}(u)$ which exchanges a and b, corresponding to an $\hat{r}'$ which reverses e' (Fig. 3).

Because of Theorem 3.6, it remains to show three facts about such situations.

(1) $\Rightarrow$ **Fact 1**: *If $\hat{G}$ is geometrizable at w and every $\hat{g}' \in \hat{G}'$ which reverses e' is of order* 2, *then $\hat{G}$ is geometrizable at u*. To see this, first notice that $\hat{r}$ is of order 2, thus equicyclic at u. If $\mathrm{stab}(u) \neq C(\hat{r}, 2)$, then there must be a non-trivial $\hat{f} \in \mathrm{stab}(u)$ which fixes b. Since $\hat{G}$ is geometrizable at w and $\hat{f} \in \mathrm{stab}(w)$, $\hat{f}^2 = \hat{id}$ so $\hat{f}$ is strictly bicyclic at u. Here $\mathrm{stab}(u) = D(\hat{f}, \hat{r}, 2)$ since anything more would require another element strictly bicyclic at w. The product of this element with $\hat{f}$ would have to fix b, thus not be equicyclic, contradicting Remark 3.5.

(1) $\Leftarrow$ **Fact 2**: *If $\hat{G}$ is geometrizable at u then every $\hat{g}' \in \hat{G}'$ which reverses e' is of order* 2. This follows immediately since the corresponding $\hat{g}$ must be equicyclic at u, whose star has only 2 edges.

(2) $\Rightarrow$ **Fact 3**: *If* $\mathrm{stab}(w) = C(\hat{g}, m)$ *where $\hat{g}$ is equicyclic at w then* $\mathrm{stab}(u) = C(\hat{r}, 2)$ *and $\hat{r}$ is equicyclic at u*. This follows because any non-trivial graph automorphism which fixes b is not equicyclic at w. □

The next two results are similar to Theorem 3.1 and Corollary 3.2 in [2].

Lemma 3.8. *Let Γ be a graph, β be a path from v to w, and $\hat{g} \in \mathrm{stab}(w) \subseteq \mathrm{Aut}(\Gamma)$. In the subgraph of Γ fixed by $\hat{g}$, the component Δ which contains w is such that* $M_\beta(\mathrm{Fix}(A(\hat{g}))) = \pi_1(\Delta, w)$.

Proof. For each $x \in F$ there is a unique reduced circuit λ_x at w such that $M_\beta(x) = [\lambda_x]$. By definition, the equality $A(\hat{g})(x) = x$ is equivalent to $[\hat{g}(\lambda_x)] = [\lambda_x]$, and thus $\hat{g}(\lambda_x) = \lambda_x$ since λ_x is reduced. If $\mathrm{Fix}(A(\hat{g})) = \langle x_1, \ldots, x_k \rangle \subseteq F$, then let $\Delta = |\lambda_{x_1}| \cup \cdots |\lambda_{x_k}| \subseteq \Gamma$. This means that $\hat{g}|_\Delta = \hat{id}$ and $M_\beta(\mathrm{Fix}(A(\hat{g}))) = M_\beta(\langle x_1, \ldots, x_k \rangle) = \langle [\lambda_{x_1}], \ldots, [\lambda_{x_k}] \rangle \subseteq \pi_1(\Delta, w)$. If there existed a $[\lambda] \in \pi_1(\Delta, w)$ such that $[\lambda] \notin \langle [\lambda_{x_1}], \ldots, [\lambda_{x_k}] \rangle$ with $\hat{g}(\lambda) = \lambda$ for the reduced circuit representative λ, then $(M_\beta)^{-1}([\lambda]) \in \mathrm{Fix}(A(\hat{g})) \backslash \langle x_1, \ldots, x_k \rangle$ would contradict $\mathrm{Fix}(A(\hat{g})) = \langle x_1, \ldots, x_k \rangle$. □

Corollary 3.9 (*Dyer and Scott [4]*). *The fixed subgroup of a periodic automorphism is a free factor.*

Proof. This follows from the previous lemma and Proposition 2.12 since $\pi_1(\Delta, w)$ is a free factor of $\pi_1(\Gamma, w)$ by the Seifert-van Kampen Theorem, and isomorphisms preserve free factors. □

Theorem 3.10. *If* $\mathbf{G} \subset \mathrm{Out}(F)$ *is finite and geometric, then any finite* $H \subset \Phi^{-1}(\mathbf{G}) \subset \mathrm{Aut}(F)$ *is either*

(1) *$H = C_m$ where all non-trivial elements have trivial fixed subgroups.*
(2) *$H = D_m$ where all elements not in $C_m \subset D_m$ have cyclic fixed subgroups.*

Furthermore, if $\mathbf{G}$ *is orientably geometric, then only* (1) *occurs.*

Proof. If $\mathbf{G}$ is geometric then by definition and Theorem 3.2, there is a punctured surface S and $\tilde{G} \subset \mathrm{Homeo}(S)$ with a $\tilde{G}$-invariant spine $\Gamma \subset S$ for which $\hat{G} = \tilde{G}|_\Gamma \subseteq \mathrm{Aut}(\Gamma)$ satisfies $\mathbf{O}(\hat{G}) = \mathbf{G}$. We ensure that $\hat{G}$ is reduced and non-edge-reversing by subdividing any reversed edges and then contracting any $\hat{G}$-invariant forests in Γ.

If $H \subset \Phi^{-1}(\mathbf{G})$ is a finite subgroup, then by Proposition 2.12 there is a path β from v to w and $\hat{H} \subseteq \mathrm{stab}(w)$ such that $H = A(\hat{H})$. Since $\mathbf{G}$ is geometric, $\hat{G}$ is geometrizable at every vertex of Γ by Theorem 3.7. Thus any non-trivial $h \in H$ is $h = A(\hat{h})$ where $\hat{h}$ is either bicylic or equicyclic at w.

Suppose h is such that $\mathrm{rank}(\mathrm{Fix}(h)) \geqslant 2$. By Lemma 3.8, there is a subgraph Δ with $\mathrm{rank}\ (\pi_1(\Delta, w)) \geqslant 2$ and $\hat{h}|_\Delta = \hat{id}$. This necessitates a vertex $u \in V(\Delta)$ of valence 3 or higher in Δ, so $\hat{h} \in \mathrm{stab}(u)$ is non-trivial and fixes at least three edges of $\mathrm{star}(u)$. Thus $\hat{h}$ is neither bicyclic nor equicyclic at u, a contradiction. It follows that $\mathrm{Fix}(h)$ is cyclic for any non-trivial $h \in H$.

If $h = A(\hat{h})$ where $\hat{h}$ is equicyclic at w, then $\hat{h}$ fixes no edges of $\mathrm{star}(w)$. By Lemma 3.8, $\mathrm{Fix}(h)$ is trivial. Thus, if $\hat{H} = C(\hat{h}, m)$ where $\hat{h}$ is equicyclic at w then (since every power of $\hat{h}$ is also equicyclic at w) we have case (1). If H is cyclically generated by an $\hat{h}$ which is bicyclic at w, then $H = C(\hat{h}, 2)$. This is case (1) if $\mathrm{Fix}(h)$ is trivial and case (2) otherwise. Finally, if $\hat{H}$ is not cyclically generated, then $\hat{H} = D(\hat{f}, \hat{r}, m) \subseteq \mathrm{stab}(w)$. If we make sure that $\hat{r}$ is a power of the generator of $\mathrm{stab}(w)$ which is equicyclic at w (only a concern when $m = 2$), then $H = D(f, r, m)$ with $\mathrm{rank}(\mathrm{Fix}(r)) = 0$, case (2). □

4. Main results

We are now able to give necessary and sufficient conditions for a finite group of outer automorphisms to be orientably-geometric, and for a finite abelian group to be geometric. Of separate interest, we prove the existence of periodic non-geometric automorphisms, as well as show that two geometric automorphisms can generate a finite non-geometric group. Also, we show how a group that is geometric but not orientably-geometric can fail to have an orientably-geometric subgroup of index 2.

Theorem 4.1. *A finite abelian subgroup* $\mathbf{G} \subset \mathrm{Out}(F)$ *is geometric if and only if every finite subgroup* $H \subset \Phi^{-1}(\mathbf{G}) \subset \mathrm{Aut}(F)$ *has one of the forms listed in Theorem* 3.10. *If* $H = D_m$, *then* $m \leqslant 2$.

Proof. The forward direction follows by Theorem 3.10. Since the restriction of Φ : $\mathrm{Aut}(F) \to \mathrm{Out}(F)$ to any finite group is injective, every finite $H \subset \Phi^{-1}(\mathbf{G})$ is abelian. If D_m is abelian, then $m \leqslant 2$.

The converse is proven as follows. The Realization Theorem 2.10 ensures the existence of a graph Γ and $\hat{G} \subseteq \mathrm{Aut}(\Gamma)$ for which $\mathbf{O}(\hat{G}) = \mathbf{G}$. As before, we can assure that $\hat{G}$ is reduced and non-edge-reversing. We will show that $\hat{G}$ is geometrizable at every vertex in Γ, hence that $\hat{G}$ and $\mathbf{G}$ are geometric by Theorem 3.7.

Let $w \in V(\Gamma)$ be any vertex of Γ and $\hat{h} \in \mathrm{stab}(w)$ some non-trivial element of the stabilizer. If β is a path from v to w in Γ, then by Lemma 2.12, $h = A(\hat{h})$ is periodic, thus $H = C(h, m) \subset \Phi^{-1}(\mathbf{G})$ is a finite cyclic subgroup. By hypothesis, either h and $\hat{h}$ are of order 2 with infinite cyclic fixed subgroups or else all non-trivial powers have trivial fixed subgroups.

If $\hat{h}^k(e) = e$ for some non-trivial $\hat{h}^k \in \mathrm{stab}(w)$ and $e \in \mathrm{star}(w)$, then since $\hat{G}$ is abelian, $\hat{h}^k(\hat{g}(e)) = \hat{g}\hat{h}^k(e) = \hat{g}(e)$ for all $\hat{g} \in \hat{G}$, that is, $\hat{h}^k$ fixes all the edges of $\hat{G}e$. Since $\hat{G}$ is reduced, $\hat{h}^k$ fixes a simple loop of $|\hat{G}e|$ which contains e. If we denote by $\Delta_{\hat{h}^k} \subset \Gamma$ the connected component of $\left\{|e| \subset \Gamma \mid \hat{h}^k(e) = e\right\} \cup w$ which contains w, then $\Delta_{\hat{h}^k}$ is a union of simple circuits (or just $\Delta_{\hat{h}^k} = w$).

If $\hat{h}$ is of order 2 with infinite cyclic fixed subgroup, then Lemma 3.8 shows that $\mathrm{rank}(\pi_1(\Delta_{\hat{h}}, w)) = 1$, thus $\Delta_{\hat{h}}$ consists of one simple circuit. It follows that $\hat{h}$ fixes exactly two edges of $\mathrm{star}(w)$, so $\hat{h}$ is bicyclic at w.

If all non-trivial $\hat{h}^k$ have trivial fixed subgroups then each $\Delta_{\hat{h}^k}$ must be trivial ($\Delta_{\hat{h}^k} = w$) by Lemma 3.8, so each non-trivial $\hat{h}^k$ fixes no edges of $\mathrm{star}(w)$. This means that $\hat{h}$ is equicyclic at w.

If $\mathrm{stab}(w)$ is not cyclic, then since $H = A(\mathrm{stab}(w)) \subset \Phi^{-1}(\mathbf{G})$ is a finite subgroup, we must have $H = D(f, r, 2)$ where r has a trivial fixed subgroup. It follows that $\mathrm{stab}(w) = \hat{D}(\hat{f}, \hat{r}, 2)$ with $\hat{r}$ is equicyclic at w as shown above. Since all other elements of $\mathrm{stab}(w)$ have been shown either equicyclic at w or bicyclic at w (all of order 2 and thus certainly bicyclic at w), it follows that $\hat{G}$ is geometrizable at w. $\square$

Corollary 4.2. *A periodic outer automorphism* $\mathbf{g} \in \mathrm{Out}(F)$ *is geometric if and only if any finite subgroup* $H \subset \Phi^{-1}(C(\mathbf{g}, m)) \subset \mathrm{Aut}(F)$ *is either* $H = D_1$ *where the only non-trivial*

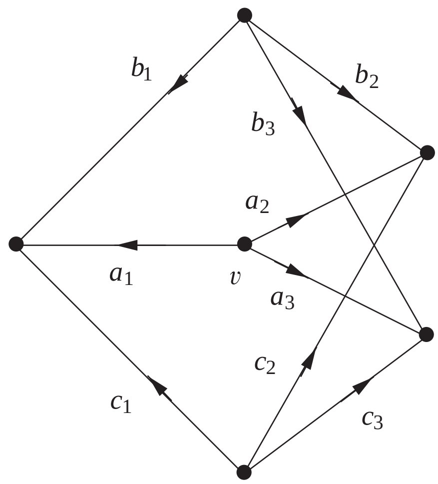

Fig. 4. Graph Γ.

element has an infinite cyclic fixed subgroup or else $H = C_m$ where all non-trivial elements have trivial fixed subgroups.

Example 4.3. Let the group $\hat{G}$ be generated by $\hat{f}, \hat{g} \in \mathrm{Aut}(\Gamma)$ on the graph Γ (Fig. 4) as follows:

$$\begin{array}{llll} \hat{g}: & a_i \to a_{i+1} \pmod 3 & \hat{f}: & a_i \to b_i \\ & b_i \to c_{i+1} \pmod 3 & & b_i \to c_i \\ & c_i \to b_{i+1} \pmod 3 & & c_i \to a_i. \end{array}$$

It can easily be verified that the stabilizer of each vertex is either cyclic or dihedral, and that no circuits are ever fixed by a non-trivial graph automorphism. Thus by Proposition 2.12 and Lemma 3.8, all of the necessary conditions of Theorem 3.10 are satisfied for $\mathbf{O}(\hat{G}) \subset \mathbf{Out}(F)$. However, since the non-trivial $\hat{g}^3$ fixes all three edges of star(v), it is clear that $\hat{G}$ is not geometric by Theorem 3.6. Since all of the edges of Γ are in one orbit, the work of Krstić [8] proves that this is the only reduced group of graph automorphisms which realizes $\mathbf{O}(\hat{G})$. Thus $\mathbf{O}(\hat{G})$ is not geometric and the conditions of Theorem 3.10 are not sufficient for non-abelian finite groups of outer automorphisms.

Example 4.4. Let $F = \langle a, b, c \rangle$ and $g \in \mathrm{Aut}(F)$ be the automorphism which takes a to a, b to b, and c to c^{-1}. By Corollary 4.2, since the fixed subgroup is non-cyclic we know that g $= \Phi(g) \in \mathrm{Out}(F)$ is not geometric. Thus there exist periodic non-geometric automorphisms.

Example 4.5. Let $F = \langle a, b, c \rangle$ and $f \in \mathrm{Aut}(F)$ be the automorphism which takes a to a, b to b^{-1}, and c to c^{-1}. It is easily seen that f is of order 2 and that $\mathrm{Fix}(f) = \langle a \rangle$, thus $\Phi(f) = \mathbf{f} \in \mathrm{Out}(F)$ is geometric by Corollary 4.2. Similarly, if $r \in \mathrm{Aut}(F)$ takes a to a^{-1}, b to b, and c to c^{-1}, then $\Phi(r) = \mathbf{r} \in \mathrm{Out}(F)$ is also geometric. These generate the finite abelian subgroup $\mathbf{G} = D(\mathbf{f}, \mathbf{r}, 2) \subset \mathrm{Out}(F)$. Since $D(f, r, 2) \subset \Phi^{-1}(\mathbf{G})$ is a finite dihedral subgroup with all fixed subgroups non-trivial, $\mathbf{G}$ is not geometric by Theorem 3.7. Thus two geometric outer automorphisms can generate a finite non-geometric group.

Lemma 4.6. *If $\hat{H} \subseteq \hat{G} \subseteq \mathrm{Aut}(\Gamma)$ is cyclic, then any edge $e \in E(\Gamma)$ fixed by an element $\hat{g} \in \hat{H}$ is also fixed by any other element $\hat{f} \in \hat{H}$ of the same order as $\hat{g}$.*

Proof. In a cyclic group, any two elements of the same order generate the same subgroup. Thus if $\hat{g}(e) = e$ for $e \in E(\Gamma)$, then $\hat{f}(e) = \hat{g}^i(e) = e$. $\square$

Lemma 4.7. *Suppose that $e \in E(\Gamma)$ and no element of $\hat{G} \subseteq \mathrm{Aut}(\Gamma)$ maps $i(e)$ to $t(e)$. Then if $a, b \in \hat{G}e$ share a vertex u, there is an $\hat{f} \in \mathrm{stab}(u)$ with $\hat{f}(a) = b$.*

Proof. By contrapositive. By definition, there exists $\hat{g}, \hat{h} \in \hat{G}$ such that $\hat{g}(e) = a$ and $\hat{h}(e) = b$ so $\hat{f} = \hat{h}\hat{g}^{-1} \in \hat{G}$ takes a to b. If $\hat{f} \notin \mathrm{stab}(u)$, then $\hat{f}$ must map u to w, the other endpoint of b. Either $\hat{h}^{-1}\hat{f}\hat{h}$ (if $\hat{h}(i(e)) = u$) or $\hat{h}^{-1}\hat{f}^{-1}\hat{h}$ (if $\hat{h}(i(e)) = w$) will map $i(e)$ to $t(e)$. $\square$

Theorem 4.8. *A finite group $\mathbf{G} \subset \mathrm{Out}(F)$ is orientably-geometric if and only if every finite subgroup $H \subset \Phi^{-1}(\mathbf{G}) \subset \mathrm{Aut}(F)$ is $H = C_m$ and every non-trivial element has a trivial fixed subgroup.*

Proof. The forward direction is immediate by Theorem 3.10. For the converse we proceed as follows. By the Realization Theorem 2.10, we can find a graph Γ and $\hat{G} \subseteq \mathrm{Aut}(\Gamma)$ for which $\mathbf{O}(\hat{G}) = \mathbf{G}$. As usual, we can assume that $\hat{G}$ is reduced and non-edge-reversing. Let w be some vertex of Γ. If β is a path in Γ from v to w, then $H = A(\mathrm{stab}(w))$ is a finite subgroup. By hypothesis, H and $\mathrm{stab}(w)$ are cyclic. If $\mathrm{stab}(w) = C(\hat{r}, m)$ then every non-trivial power $\hat{r}^k$ has $\mathrm{Fix}(A(\hat{r}^k))$ trivial. Lemma 3.8 implies that $\Delta = \{e \in E(\Gamma) \mid \hat{r}^k(e) = e\}$ has $\pi_1(\Delta, w)$ trivial, thus $\hat{r}^k$ fixes no non-trivial circuits through w.

We will show that each non-trivial $\hat{r}^k$ fixes no edges of $\mathrm{star}(w)$, thus $\hat{r}$ is equicyclic at w. We will do this by showing that if $\hat{r}^k$ fixes an edge of $\mathrm{star}(w)$, then it fixes a non-trivial circuit through w. By Theorem 3.7, it will follow that $\hat{G}$ and $\mathbf{G}$ are orientably-geometric.

Suppose there is an edge $e \in \mathrm{star}(w)$ with $\hat{r}^k(e) = e$. We will separate two cases as to whether or not there exists a $\hat{g} \in \hat{G}$ with $\hat{g}(i(e)) = t(e)$.

If there is a $\hat{g} \in \hat{G}$ with $\hat{g}(i(e)) = t(e)$, then $\hat{g}^{-1}\hat{r}^k\hat{g} \in \mathrm{stab}(w)$ and has the same order as $\hat{r}^k$. By Lemma 4.6, $\hat{g}^{-1}\hat{r}^k\hat{g}(e) = e$ thus $\hat{r}^k(\hat{g}(e)) = \hat{g}(e)$. This implies $\hat{g}^{-2}\hat{r}^k\hat{g}^2 \in \mathrm{stab}(w)$, which is of the same order as $\hat{r}^k$, so again by Lemma 4.6, $\hat{g}^{-2}\hat{r}^k\hat{g}^2(e) = e$ and $\hat{r}^k(\hat{g}^2(e)) = \hat{g}^2(e)$. Continuing this, we see that $\hat{r}^k$ fixes $\lambda = \{\hat{g}^m(e) \in E(\Gamma) \mid m = 1 \ldots (\text{order of } \hat{g})\}$. We have $t(\hat{g}^i(e)) = i(\hat{g}^{i+1}(e))$ for each $i \in \mathbb{N}$ and $\hat{g}^{i+1}(e) \neq \overline{\hat{g}^i(e)}$ since $\hat{G}$ is non-edge-reversing, so λ is a non-trivial circuit at w which is fixed by $\hat{r}^k$, a contradiction.

Now suppose there does not exist a $\hat{g} \in \hat{G}$ for which $\hat{g}(i(e)) = t(e)$. Let $a \in \hat{G}e$ be some edge for which $\hat{r}^k(a) = a$ and take $b \in \hat{G}e$ to be any edge which shares an endpoint u with a. By Lemma 4.7, there is an $\hat{h} \in \mathrm{stab}(u)$ for which $\hat{h}(a) = b$. Since $\mathrm{stab}(u)$ is cyclic, as are all vertex stabilizers in $\hat{G}$, and since $\hat{r}^k \in \mathrm{stab}(u)$, then $\hat{r}^k(b) = \hat{r}^k(\hat{h}(a)) = \hat{h}\hat{r}^k(a) = \hat{h}(a) = b$. Since $\hat{r}^k$ fixes $e \in \hat{G}e$, it follows that $\hat{r}^k$ fixes the entire connected component of

$\hat{G}e$ containing e. Since $\hat{G}$ is reduced, there is a simple circuit including e in this component, thus $\hat{r}^k$ fixes a non-trivial circuit through w, a contradiction. □

It may seem intuitive that if a geometric group of outer automorphisms is not orientably-geometric then it will have an orientably-geometric subgroup of index two, but the following example shows that this is not necessarily the case.

Example 4.9. On the free group $F = \langle a, b, c\rangle$ let f be the automorphism that takes a to a, b to b^{-1}, and c to $bc^{-1}b^{-1}$, and let r be the automorphism that takes a to a^{-1}, b to b^{-1}, and c to bcb^{-1}. Here $\mathbf{G} = \Phi(\langle f, r\rangle) = \mathbf{D}(f, r, 2)$ is abelian. It can be seen that $\mathbf{G}$ is geometric by Theorem 4.1 since the fixed subgroup of each automorphism is at most infinite cyclic, and since every subgroup of order four in $\Phi^{-1}(\mathbf{G})$ (for example $\{1, f, r, fr\}$) has a non-trivial element with a trivial fixed subgroup. The group $\mathbf{G}$ is not orientably-geometric, however, by Theorem 4.8 because $f \in \Phi^{-1}(\mathbf{G})$ is of order 2 and fixes $\langle a\rangle \subset F$.

The subgroups $\Phi(\langle f\rangle)$ and $\Phi(\langle fr\rangle)$ are obviously not orientably-geometric since f and fr fix$\langle a\rangle$ and $\langle b\rangle$, respectively. The last remaining non-trivial subgroup $\Phi(\langle r\rangle) \subset \mathbf{G}$ is also not orientably-geometric because of $\mathrm{Inn}_{b^{-1}} \circ r \in \Phi^{-1}(\Phi(\langle r\rangle))$. This automorphism takes a to $b^{-1}a^{-1}b$, b to b^{-1}, and c to c, thus is periodic of order 2 and fixes $\langle c\rangle$.

Thus this $\mathbf{G}$ is a geometric group of outer automorphisms that is not orientably-geometric and does not contain an orientably-geometric subgroup of index 2.

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References

[1] M. Bestvina, M. Handel, Train tracks for surface homeomorphisms, Topology 73 (1995) 109–140.
[2] M. Culler, Finite groups of outer automorphisms of a free group, Contributions to group theory, 197–207, Contemporary Math., 33, Amer. Math. Soc., Providence, RI, 1984.
[3] W. Dicks, E. Ventura, Irreducible automorphisms of growth rate one, J. Pure Appl. Algebra 88 (1993) 51–62.
[4] J.L. Dyer, P.G. Scott, Periodic automorphisms of free groups, Comm. Algebra 3 (1975) 195–201.
[5] A. Karrass, A. Pietrowski, D. Solitar, Finite and infinite cyclic extensions of free groups, J. Austral. Math. Soc. 16 (1973) 458–466.
[6] S.P. Kerckhoff, The Nielsen realization problem, Ann. Math. 117 (1983) 235–265.
[7] D.G. Kramtsov, Solvability of the conjugacy problem for finite subgroups in automorphism groups of free groups, Algebra Logic 34 (1995) 311–337.
[8] S. Krstić, Actions of finite groups on graphs and related automorphisms of free groups, J. Algebra 124 (1989) 119–138.
[9] J. Los, Z. Nitecki, Embedding Groups of Graph Automorphisms in Surfaces, Topology 43 (2004) 49–69.
[10] W. Magnus, A. Karrass, D. Solitar, Combinatorial Group Theory, Interscience, New York, 1966.
[11] W. Mangler, Die Klassen topologisher Abbildungen einer geschlossenen Fläche auf sich, Math. Z. 44 (1939) 541–554.

[12] J.P. Serre (transl. J. Stillwell), Trees, Springer, New York, 1980.
[13] J. Stallings, Topologically unrealizable automorphisms of free groups, Proc. Amer. Math. Soc. 84 (1982) 21–24.
[14] H. Zieschang, E. Vogt, H.D. Coldewey (transl. J. Stillwell), Surfaces and Planar Discontinuous Groups, Lecture Notes in Mathematics, Vol. 835, Springer, Berlin, 1980.
[15] B. Zimmermann, Über Homöomorphismen n-dimensionaler Henkelkörper und endliche Erweiterungen von Schottky-Gruppen, Comment. Math. Helv. 56 (1981) 474–486.